# Mathematical Maps –
# Students Working With Geography as a Global Metaphor for Mathematics

**Håkan Lennerstad**
Blekinge Institute of Technology

*In mathematical map making, a group of students discuss and formulate the mathematics they know in order to form an overall picture, usually as a geographical map. The design of the geography is designed by the student group to reflect mathematical ideas – their importance, character and connections with each other. It is possible to observe that students show a remarkable ease and energy in formulating mathematical ideas in this form. Conclusions may be that geography/landscape is a context that is extremely well known for everyone, and thus provides a rich source for metaphors. Geography is a global metaphor in the sense that it serves as a metaphoric scene for all of mathematics, providing countless "local" metaphors within this scene. Indeed, natural language already contains a large number of geographical metaphors for mental activity. Mathematical maps have been made at all levels, from kindergarten to university.*

## Introduction

Researchers have persistently emphasized the fundamental importance of mathematical student communication for conceptual understanding. Mentioning one example only, Jodie Hunter starts her paper (Hunter 2009) as follows:

> Developing student communication of productive mathematical reasoning has become a key objective for teachers in Western mathematics classrooms of the 21st century (Walshaw & Anthony, 2008). The pedagogical intent is that students are involved in learning communities in which all participants have opportunities to engage in productive mathematical discourse (Manoucheri & St John, 2006).

With this purpose, to develop student communication about mathematics, the activity of mathematical map making is presented in this paper. Thus, the following questions are central for this paper: How can we achieve active discussions about mathematical ideas among students? How can we make students aware of their own mathematical knowledge? How can students benefit from each other's mathematical knowledge?

This paper is describes mathematical map making, and how this activity possibly may provide partial answers to the questions above. I investigate the question: What is the effect of mathematical map making in mathematics education?

To this date and to my knowledge there exist about fifty mathematical maps. All are made during the last eight years, and all are made in Sweden, except one map: "Calculus in One Variable". This map is a translation and completion of the first map from 2002, which appeared in the calculus text book *Envariabelanalys med dialoger* (Lennerstad, 2002). The paper *The scope of geography as a metaphor for mathematics – a case study* (submitted as short communication to MADIF 7) is devoted to this particular map, to describe the depth that is offered by the geographical metaphor to describe mathematics. These two maps cover a basic university calculus course in one variable. Since then, student groups have made maps in all levels of the educational system.

The organization of this paper is as follows. This introduction gives a brief description of intentions and practice of mathematical map making. After this presentation a comparison to previous work is much easier to do, which is Section 2. Section 3 presents empirical information about map making activity. Section 4 contains a discussion, and Section 5 describes further developments.

**What is a mathematical map?**

A mathematical map is usually a landscape with rivers, cities, mountains lakes and oceans, or a city with streets, buildings, industries, rivers and parks and so on. All names on the map are mathematical words, concepts, numbers or formulas. It may contain everything that appears on real maps, at the discretion of the group. The anatomy of the landscape or the city is chosen and designed by the group/person, in order to describe how mathematical concepts and mathematical activities to the group's/person's knowledge are related. Thus, mathematical issues are suggested, negotiated and drawn by the students during the activity. The learning that is implied in this activity is the core of map making. Examples of mathematical maps can be studied at the web site www.bth.se/matematikkartor (in Swedish).

This approach allows students to express their subjective experiences of mathematics in the same framework as pure mathematical relations. For example, mountain areas may represent troublesome calculations, high altitude for abstractness, and a cemetery for abandoned mathematics.

**Global metaphors**

Usually geography has played the role as a global metaphor in the activity that has occurred. By "global metaphor" we mean not only isolated metaphorical images for ideas in the subject (mathematics), but that *the subject is described within an entire metaphoric scene*. Thus, a large number of "local metaphors" appear within the "scene" provided by the global metaphor. An example of a local geographic metaphor, produced by students, is naming a highway as a Calculator highway, due to the speed allowed. It is an example of a "local metaphor", being a part of the geographic global metaphor.

Another global metaphor, which is not geographic, has been presented in an art exhibition (Håkan Lennerstad: *Iconic Mathematics,* Blekinge Museum, Sweden, October 2008). Here the periodic system of the elements plays the role as a global metaphor for a selection of 100 of the most basic and important functions of one variable. Within this metaphor many important properties of functions are easily represented, such as odd, even, continuous, convex, concave, increasing, decreasing, maximums, minimums, inflection points, singularities, inverse functions, inverted values, derivatives, translates, dilates, Fourier transforms and Laplace transforms.

**Overall purposes**

The overall purpose of mathematical map making is to develop mathematical understanding by facilitating authentic mathematical dialogue, building a language for mathematical dialogue and increase students' metacognition in mathematics, all based on the students' own mathematical experiences. The purpose is to open a possibility, in the view of the students, for mathematical dialogue by allowing an extended language about mathematics to develop.

**How has the activity been organized?**

Most common is that the work has been done in the following phases.

1. *Mathematical words collecting*. Before forming groups, all students participate in suggesting mathematical words, which may be collected on the blackboard. This may require about half an hour or more, and may give from 20 to hundred words, including discussions about which words are mathematical. Certainly, words may appear and be included in the map later.

2. *Group forming*. In the next phase, students form groups. There have been groups of sizes from one to eight. In general, smaller groups are recommended for more active students. On the other hand, less active students may prefer smaller groups, and in some large groups all students have been very active, later in different parts of the map.

3. *Choice of overall structure*. This phase consists in deciding an overall organization of the map. It can be started by choosing subsets of the collection of mathematical words, a task that can be done in many ways, and then consider each subset as a country or other province. There are alternatives to geographic maps. One group has chosen bottles with mathematical labels standing in a shelf, while another choose a fruit tree. Furthermore, a geographical alternative can be carried out in many different ways, from maps of continents to one single apartment. The groups may be reformed after this stage if not everyone is in tune with the basic idea of the group.

4. *Filling the structure with mathematical knowledge*. This is the main phase of the work. If the overall structure is well chosen and relevant for the persons involved, ideas usually flow, and very much of the mathematics that the students are aware of finds ways to be represented in connection to other topics. This is a negotiating activity about mathematical concepts. It is often done using post-it-notes, allowing flexibility.

5. *Drawing, painting and completing*. This part is mostly artistic, although mathematical completion or correction may occur also here, within the frames which now are rather rigid. Ideas of different mathematics maps, based on other grounds, may emerge, to be done later.

6. *Presentation to other groups*. The group presentations provide overall mathematical structure and details, seamlessly with the student's personal activity, engagement and cooperation.

Although mathematical facts and relations are at the core in the activity, a mathematics map is necessarily also subjective. The aim is bridging subjectivity and mathematical objectivity. Therefore, the result is mathematical, but also a work of art, by the group.

**Why geography?**

How do the two domains mathematics and geography relate? We mention here three fundamental arguments for the use of geography as a metaphor for mathematics.

- *Geographical metaphors for ways of thought* are very common in natural language. We talk about *ways* to find a solution, of scientific *fields*, *stormy* relations and *surveys* of ideas. Stocks may *skyrocket*, people may face *oceans* of work. The first chapter in the volume *The mathematical experience* (Hirsch/Davies 1998) is indeed titled *The mathematical landscape*.
- Landscapes/geography is fundamental for human life, at least in the meaning of surroundings to the home. Maps may represent nature, geology, politics, energy, economics, demographics, and much more. We practice and develop knowledge of this context every time we transport ourselves. Therefore, *landscapes/geography is an extremely well known and flexible context, able to represent facts and relations within a vast spectrum of aspects*.
- Landscapes/geography is *very well known for children*. It provides a substantial part of their concrete knowledge of the world. Geography can for smaller children mean the home, the neighbourhood, playground and school, or the house or apartment where the child lives.

Here the reasons can be found for the observation that student seem to almost always find an interesting way to represent a mathematical idea geographically.

The two contexts are very different in many respects: mathematics being abstract, while geography is very concrete. This difference provides actually the goal of the activity, because *student's learning of mathematics is in need of bridging students' existing concrete experiences to the abstractness of mathematical ideas*. Mathematics in geographical form is abstractness in concrete form.

Concrete metaphors are here seen as preliminary tools for thinking about, handling and coming to terms with abstract concepts. It is important that the limits of each metaphor become visible, so that the understanding may mature by the aid of other metaphors into a fuller and in the end less metaphor-dependent mathematical knowledge.

Is map making shallow in the sense that after some work it is saturated – and more relevant mathematical knowledge cannot be represented? The companion paper *The scope of geography as a global metaphor for mathematics – a case study* (Lennerstad 2010) tries to answer that question by showing that the geographical metaphor can be deepened to represent rather advanced properties of mathematics.

## Related concepts and research

One of the most famous approaches for representing theory in a graphical way is concept mapping, which was developed mainly by Joseph D. Novak (Novak 1985). Concept mapping is based on a constructivist view of learning, such as that of David Ausubel (Ausubel 1968), who stresses the importance of that students construct their knowledge, as well as the importance of prior knowledge:

> The most important single factor influencing learning is what the learner already knows. Ascertain this and teach accordingly.

A concept map is a flexible means to formulate basic relations in a subject. The anthology *Concept mapping in Mathematics* (Afamasaga-Fuata'i 2009) appears to be the first comprehensive book on concept mapping applied to mathematics learning. In (Grevholm 2005), work with concept mapping is described for teacher students' learning in mathematics and mathematics education. Note that geography plays no role in concept mapping.

Mind mapping (Buzan 2000) is related to concept mapping. It is usually non-geographic, less regulated/structured than concept maps, and thus perhaps more suitable for brainstorming where limiting prohibitions are temporarily avoided. A development of mind mapping for logic is logical graphs (Lennerstad 1996A and 1996B). Here a statement is never repeated, instead implication arrows are drawn from a single occurrence, which makes the logical structure of an argumentation obviously visible. The presentation form is consistent with the logic.

Geographical metaphors have been used in some contexts, such as *Tillvarons atlas* (von Swaaij & Klare, 2001), which depicts human life, and *Der atlas des managements* (Cairner, Derlove 2005), for management.

## Importance of a global metaphor

Concept mapping, mind mapping and logical graphs can be very fruitful, but are rather abstract for students to work with. A main point of this paper is that, from students' point of view, the geographical metaphor provides energy and enthusiasm to the work. When students find a fitting local metaphor it is for them both an emotional and an intellectual event. This is very important for the emotional energy of their work.

Maria Selander was the first teacher that tried mathematical map making in a student group (Selander & Lennerstad 2004) in Strängnäs, for 16 year old students. The work took five weeks with about four hours work in school each week. To be able to finish the map, at least one group worked extensively also after school. She observed that students that do not have top grades often were most active, and that the map drawing students became more active during the "normal" mathematics classes. Initially the activity was time consuming for her as a teacher, but not after that phase.

Mathematical map making has taken place in the teachers' education in Linköping University and other places. In a high school in Åmål a very decorative mathematical map was made as a wall painting.

## Classroom activities and teachers' experiences

This section contains ideas and conclusions by teachers. Some of the maps can be studied at the web site www.bth.se/matematikkartor (in Swedish). The majority of the maps have been done by groups of students, while a few are done by groups of teacher student's.

A different use is map construction by a group of teachers. Even more for teachers, it appears, map drawing is a practice in mathematical dialogue, challenging the ability to listen and compromising concerning mathematics.

### Teachers' evaluations

Next follows sixteen teacher/ teacher groups that organized mathematical map making in their student groups. The teachers did the work as a part of a course in mathematics education. The excerpts are very short, focusing mainly how the work was started, main problems and overall evaluations by students and teachers.

Initially, the teachers listened to a lecture by Håkan Lennerstad, who described the idea. This meeting was three hours long and included an activity where the teachers in groups tried to find geographical metaphors for mathematical concepts. The meeting was compulsory for the teachers in their course. The following are translations of the teacher groups own words.

1. The task could be too large. The goal was to let the students, in 6$^{th}$ grade, understand how much mathematics they understand. Some made a map over the Swedish language and some over mathematics. The task was fruitful both for

teachers and students. It is a very good way for students to get to know their understanding in the subject. The students became engaged and many discussions about mathematics occurred spontaneously. Some students were frustrated over that the semester ended so they could not continue and develop their map.

2. We had some problems with the order in the class, and the students were not so motivated. There were interesting mathematical discussions, but we have that anyway. The students liked to draw, but they could likewise have done that in the art class.

3. The students immediately started to sketch their mathematics map. They used the math book to get ideas. The groups were allowed to make suggestions to maps to other groups. The cooperation in the groups has worked out very well. The students liked the project. They say that they have been able to get a better perspective for mathematics. I found the students very engaged in their work and I have seen that several students have grown during this work.

4. These students had some problems in their math learning. They made bottles with different mathematical concepts, and the amount of content reflected their knowledge of that concept. As a teacher I could observe that the student thought they knew more than they did, for example about common words as term, difference, etc. There were interesting discussions about the meaning of different mathematical words.

5. I thought a lot about how to introduce the idea to engage all of them without steering too much. They choose to make a mathematics map in the shape of a turtle, a town and a shop. Many debates appeared. For example, most of them thought that a ruler has nothing to do with mathematics, but had eventually to agree by a convincing argumentation by one girl. I got a deepened knowledge about how students think. Another large gain is that they have powerful argumentation to add something mathematical on the map. We can reach the students very much better if we are familiar with their mathematical thoughts.

6. Most of the student groups made landscapes, but one group, for example, made a mathematical soccer field. We think it worked well especially as a tool for the students to construct an overview and to concretize their own view of how different parts of mathematics are related.

7. We started the work by presenting the idea of map drawing and explaining that we for this need mathematical words. After the students had suggested words we grouped them in natural sets according to the students' opinion. The students became a bit impressed by the large number of mathematical words that they understand. The work with the maps took four lessons, and almost all groups showed a miraculous imagination when they had understood what to do. Some groups, however, had difficulties in deciding over a common basic idea for the map.

8. Thirty 6th graders were involved in the project, and after having collected a large number of mathematical words the mathematical debates started. It was very interesting to walk around and listen. Thy really thought about the meaning of the mathematical words. Some groups had problems to cooperate. The artistic and the verbally fluent students liked the drawing of the map. They liked it, they learned the connections in mathematics, two of them said that they learned what algebra is, and a few found map making boring.

9. The groups were grades 4 and 5, and the work started by collecting mathematical words. They made one of five areas: geometry, the quattuor species, the positional system, weight and length, and finally time. The work was concluded with a presentation in a large group. The cooperation in the groups worked well, and the fantasy and energy was flowing. It was a very nice type of work for both teachers and students.

10. In a fourth grade we collected mathematical words, and the students started with their maps in groups of two in each. The student liked the work, but had difficulties in connecting the mathematical words with concepts, they did not have so much underlying idea behind their naming with mathematical words. However, they found several nice ideas in their maps. We all liked it and will perhaps try this again in a somwhat different way.

11. We tried to make mathematics maps in a group of ten students in eighth grade who need special support. The result was very mixed. It seems like mathematics maps do not fit so well since the students lack too many basic abilities in mathematics.

12. I tried to introduce mathematics maps in a seventh and one eight grade. I tried to imagine a walk through the woods in the area and the students need to pack abilities that one need during such a walk. I am not sure about the result.

13. We had only two students who decided to make a map about addition, subtraction, multiplication and division, one in each corner of the paper. Each was an island with smaller islands around connecting to other mathematical words. They discussed very much about meaning in mathematical words and when they use them in everyday life.

14. Students in fourth and sixth grade were very diverse in mathematical knowledge. We decided that most important would be that they talk mathematics. They did not draw maps but discussed which kind of mathematics that appeared at real maps that we gave them. The students discussed mathematics very much, so we succeeded with our goal.

15. We started the project in three classes, grades four, five and six. All liked the task and were very creative. They discussed mathematics in a logical way, which was the intention.

16. I had a group with six students with extra needs. To have mathematics, I said, we need to go to the mathematics shop and buy what we need, but I do not

mean pencils and paper but words, signs and other that we use. What comes in your mind? A vivid dialogue started where many mathematical words came up on the white board. I am impressed by the number of concepts they put forward and their understanding of their connections. Particularly interesting was the discussion over numbers contra digits. The students liked the task. Mathematical map making is a good idea, I think. With the new students starting next year I intend to form groups working with mathematical maps.

**Summary of teacher comments**

Half of the teachers found that the students engaged in good cooperation or interesting discussion. Many also found that the students became more aware of their knowledge in mathematics and developed their perspective. The fantasy and energy were flowing, which is somewhat remarkable taking into account that it concerned inquiry in mathematics. A few groups had difficulties in finding a basic idea, while others became occupied in the drawing task only. Some groups where the students lack basic knowledge were successful, some were not. Note that the task was mandatory for the teachers, so some teachers were not motivated from start.

## Conclusions and developments

Mathematical dialogue and discussion are of paramount importance for mathematics understanding, expertise in mathematics education do not hesitate on this. The dominant goal of mathematical activity in school is to solve mathematical problems by calculation. Tests consist only of solving mathematical problem by calculation, or are strongly dominated by it. Many students are therefore *extremely focused towards calculation*. With such a focus, discussion and dialogue is relevant only in case of problems during calculation. As soon as dialogue opens new doors for calculation, the dialogue is abandoned and calculation is reassumed. This gives *broken dialogues.* The goal is not understanding, the goal is calculation, which of course is important, but perhaps too limited.

Broken dialogues are not typical during mathematical map making sessions. Conceptual dialogues may continue following routes of discovery – they are not suddenly abandoned and replaced by calculation. The task is completed when most of the mathematics familiar to the students in the group is represented in the map in an acceptable way.

*Is map making time consuming?* As described in Section 3, the task may be given everything from one afternoon to several weeks. A mathematics map may grow gradually during the entire education. It may be time consuming initially for a teacher who has not tried the work before – it is a different mathematical activity. The students are usually driving the activity themselves, often outside of school hours.

*Is map making useful for development of teacher skill?* The map certainly reflects the prevalent mathematics views in the student group, and can be studied with this aspect by a teacher. Listening to students' mathematical dialogues is valuable. It is also possible to develop questions and deepen the mathematical content of the dialogue. This may sometimes be important since there always is a risk of a superficial map, for example only sorting words and concepts in groups/countries with no cross connections.

*What does map making give for students?* Students become more aware of the terminology in mathematics, they discuss meaning of mathematical words, and they form an overall picture of mathematics. They learn from each other and become more able in talking mathematics.

Many teachers approved of the ease with which students communicated conceptually about dialogue. Some teachers have remarked that they noticed a higher activity in normal mathematics classes for "map students". Surely a potential language for mathematical communication has been evoked, but how this is taken advantage of depends on the future activity in the classes.

Map making contributes clearly to the interest in mathematics and to the realization that mathematics contains questions that can be discussed in interesting ways. This make map making valuable in teachers' education. Further investigation is needed to evaluate long term effects.

*Development 1 – conceptual study of students' dialogue:* Students' mathematical conceptual dialogues can certainly be studied, developed and deepened by teachers, for example in cooperation with mathematicians and mathematics education researchers.

*Development 2 – teachers' and mathematicians' maps:* Experiences hint that it is not easy for a group of mathematics teachers and mathematics researchers to make a common map. Compromise about the mathematics picture is needed, which is a question rarely discussed. We here thus have a starting point for a conceptual understanding of mathematics.

*Development 3 – subject maps:* Mathematical maps may easily be generalized to "subject maps" – other subjects may be described with the geographical metaphor. Music map making has been made three times during a chamber music festival (Lyckå Kammarmusikfestival, 2007, 2008, 2009) by young musicians, aged 13 to 17. Here four music student groups with four students in each have completed a four maps in three hours with no preparation in advance. The students have expressed that they have found this work unexpectedly interesting.

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