Completing Mathematics by Teacher and Student Reflection

Håkan Lennerstad,
Blekinge Institute of Technology

Abstract

The value of mathematical dialogues between students and teachers is not restricted to the educational value for students. They are also fundamental for the teachers' ongoing professional development. This aspect is also connected to students' sense of responsibility. If the students are aware of that their mathematical activity is regarded as potentially valuable for the teachers' development of the education, and thus for the benefit of future students, this may increase the sense of significance for their own learning. Furthermore, student activity, mediated by teachers, may be used to complete "official" mathematics.

The goal of this paper is to bring together dialogue, student thinking, teacher development, and the dialogue seminar reflection tools aiming towards an extended version of mathematics – extended in conceptual direction. Reflection on student-teacher practice may have a natural output as "reflective mathematics", completing formal mathematics. The formulation of reflective mathematics is intended as a remedy of the common and reasonable complaint by students that mathematics is meningless and fragmented.

Introduction

A teacher gradually accumulates teaching experience during years of teaching. However, a large volume of teaching is not equivalent to high professional skill. Development of skill is strongly dependent on the reflective attitude towards the ongoing practice. How could activities that develop teachers' reflective attitudes in a long term perspective be designed? On which fundamental view on education could such activities be built in order to be stable and consistent?

These questions have a very wide scope and many possible answers. The fundamental view on education that is focused here describe students as not only learners, but also as authentic producers of mathematical thinking with potential value for teachers' view of mathematics, and as representatives of the present student culture. Attitudes towards mathematics itself are also important for a reflective practice, for example whether it may be reinterpreted by student thinking and student activity or not. We start by discussing action research as a reflective method. Last in the paper present the dialogue seminar for this purpose after having defined and discussed reflective mathematics.

Action research focuses teachers' development by evaluating their own practice. It is described in Miller and Pine (1990) as

> An ongoing process of systematic study in which teachers examine their own teaching and students' learning through descriptive reporting, purposeful

> conversation, collegial sharing, and critical reflection for the purpose of improving classroom practice. (p. 57)

Action research is often carried out as a collaboration between researchers at a university and teachers at an elementary school. The terms "researcher" and "teacher" has a different flavour in action research in that teachers here are considered as the primary researchers. Researchers and teachers together document the school activity in different ways, and successively make evaluations and adjustments. As we shall see later, the dialogue seminar has more the form of teacher-to-teacher meetings. Action research has often reported good improvements both in students' learning and teacher skill. It seems however as if action research not so often formulates students' views of the subject – how the general image of the subject, as mathematics, can be improved. Action research appears to be a pedagogic method primarily aimed at the improvement of the classroom practice and not so much at developing the way the subject is described. However, the method of action research can certainly also be used in this direction.

Student-teacher dialogues are two-sided, but the attention on dialogues has been largely one-sided – focusing on student learning but not much on teacher learning. It would be valuable if teachers' long term professional development also is an articulated purpose. This is the perspective described for university teaching by Huber and Hutchings (2005). They propose the development of the scholarship for teaching and learning in order to improve the quality of education. They advocate teachers to successively and explicitly learn the learning processes of students during their teaching and to formulate and discuss teaching methods and educational results with colleagues. Publishing in the area is a natural part of the development of this scholarship.

We here propose a further step in this direction, not only to students' learning but also to their knowledge, i.e. no only focusing students' reaction to the subject knowledge that is taught. It is consistent to this wider view to engage students as partners for the goal of achieving a successful education. It is naturally connected to responsibility.

The one-sidedness of the attention to student-teacher dialogues makes students into pure consumers of education, and not producers. Students' sense of responsibility in their own education is of course important, and is strongly related to the actual teacher-student relationship. However, the intention here is not to increase the formal demands on students. The intention is that students are seen as thinking persons, whose mathematical work can be important for others, not only for themselves. What students are asked for, or invited to, is to try to formulate sincerely their mathematical attempts and thoughts, and engage in dialogue with teachers and/or other students.

Thus, students are asked for, and need to be stimulated into, is fearless formulation of their mathematical thought and pondering. What is it that teachers are asked for? Again it is fearless expression of mathematics, but teachers need also to find ways to discuss and formulate their teaching practices with other teachers and with other expertise. Action research is one way. We focus here the dialogue seminar, which is an established tool for reflective practice. Here organized dialogues among teachers are seen as the main tool for formulating and extracting knowledge.

In this paper we start by discussing the value of dialogue and linguistic problems in mathematics. Then university calculus course is taken as an example of how student activity can systematically affect more than the teacher who meets the students – here it is a mathematics text book. In the next section the nature of reflective mathematics is discussed and exemplified. This is needed since the idea of reflective mathematics conflicts with the prevailing idea of mathematics as utterly unchangeable. Finally the dialogue seminar is presented, including a case study where mathematics teachers participated.

Dialogues and mathematical linguistics

The importance of dialogues in learning is well established. Johnsen Höines (2004) describes very clearly that the *differences* in view between persons engaged in a dialogue is the energy driving the ongoing mutual discovery which is typical for a dialogue, here by citing (Dysthe, 1999):

> Without the differences the interaction would not have any function. The understanding would not develop. *Different voices are not enough to create meaning; the tension and struggle between them create understanding* (Dysthe, 1999).

In the dialogue seminar there is also an explicit recognition of the authentic differences that may exist and may appear in a dialogue (Berg, 2005). The point of a dialogue is not to reach a common conclusion, so such an expectation is inappropriate.

The lack of recognition of student-teacher dialogues for teachers' development reflects teacher educators' view of student teachers, and mathematicians view on mathematics student teachers. Relations in classrooms propagate from teacher education to school. If teacher educators do not recognize their learning when teaching student teachers, one cannot expect student teachers to recognize their learning as teachers when they meet students after teacher education. Increased contact areas are called for between at least three cultures: mathematicians', teacher educators' and students'. The dialogue seminar has often been used for culture-bridging purposes.

However, increased contact areas between cultures are not without risk. When two cultures meet, both cultures are to some extent jeopardized. This requires a mutual respect, not only between individuals. In particular, the mathematics culture can be seen as deviant and fragile, and may lose important characteristics if these are not clearly recognized. One way this can happen is that a mathematics teacher, when facing the depths of some students´ difficulties, may dismiss important mathematical ideas. There is a risk of inventing less general versions of concepts that solve immediate problems, but result in more problems in the future. This is not to say that the teacher should not negotiate with students about teaching methods.

The articulation of mathematics from drawing conclusions from the stories of mathematics activity is in this paper called “reflective mathematics”. The term stands for everything that gives meaning and insight to formulas and their manipulation, and explanations that contribute to making calculations predictable. Both need to be valuable for more than a few persons. Without reflective mathematics, the subject is meaningless and unpredictable formula manipulation. Students can contribute significantly to the construction of reflective mathematics, mediated by teachers. But reflective mathematics is not only effective metaphors found by students. It may involve fundamental different views of mathematics that makes the subject more available, formulated by teachers, but perhaps originating in teacher practice. An example of this is formulated later.

By often articulated student difficulties one may say that reflective mathematics today has a weak position in mathematics. This is related to aspects of its linguistic character, which we next turn to. To illustrate this, let us compare the activity of a mathematics teacher to that of a chemistry teacher. A chemistry teacher uses words and argumentation to explain properties and reactions of chemical compounds. A mathematics teacher uses words and argumentation to explain mathematical argumentation. Note that in mathematics, the teacher method (argumentation) and the subject matter (mathematical argumentation) are both words. Now, if the mathematics teacher becomes very familiar and articulated in the mathematical argumentation, and the student feedback is weak, there is not much difference for the teacher between the two types of argumentation. Mathematical argumentation may become enough. The situation may be summarized in the following short dialogue that may follow a long uninterrupted teacher presentation.

Student: Ok… and can you now explain it?

Teacher: That is what I just did!

What is lacking is reflective mathematics. The situation in the dialogue correspond in the chemistry context to that the teacher demonstrates reactions and behaviours of chemical compounds without a word of comment.

Mathematics without reflective mathematics can be seen as a mute mathematics, although it is not silent. Mathematics is almost entirely a linguistic practice.

The event that mathematics argumentation replaces "argumentation from the outside" as described above, relies on basic properties of languages. Language users are normally not conscious of the language used since we usually focus the content we talk about, and not the language itself. M. S. Smith (1994) writes: "*In most normal everyday language use, we are not especially aware that we are following rules. We even select many of the words unthinkingly. When saying "he was kissed" we do not consciously refer to a passive rule for constructing the passive sequence. We are more concerned with expressing our thoughts and understanding what people are saying.* "

However, the language can be made visible. M. S. Smith continues:

"*It is possible, however, to shift our attention to the sounds, letters, words and constructions we are using. If, for example, someone suddenly asks a question such as:*

'What is the word for an animal you keep at house?'

'What words did she actually use when she refused?'

'What is another way of saying "I don't mind if I do"?'

then the listeners' conscious attention is directed suddenly to the language itself, and not just to meaning and messages. We could call this going into the meta mode."

"Meta mode" is equally important for becoming conscious of the linguistics of the symbolic language of mathematics. Bakhtin (1981) underlines the need for different languages to be able to see a language: "*Languages throw light on each other: one language can after all see itself only in the light of another language."*

See also Lennerstad, Mouwitz (2004) for further description about "Mathematish". This term denotes the symbolic language of mathematics seen as a language, in comparison with other languages. To summarize: partly due to the properties of languages, mathematical activities easily become pure linguistic practices – argumentation "from the inside". Then authentic views on mathematics from students and teachers disappear, which is the disappearance of reflective mathematics.

A university project for student influence

The project "Student influence of text books" is a starting point of this paper, see Lennerstad (2005a) and Lennerstad, Erman, Samuelsson (2006). It was funded by the Swedish Council for Higher Education. Students on a calculus course at undergraduate level were able to post their mathematical questions and comments on a web page. Teachers and graduate students answered the questions. The aim was twofold: the obvious one of helping the students, and the less obvious one: to use the communications to improve the text book used, which was Lennerstad (2002).

The questions were stated in relation to the text book. The author studied the questions afterwards, and made several changes as a result of this. The changes in the book were not vast, but noticeable. The book is now printed in a new version, Lennerstad (2005b), including the student-inspired revisions.

Initially, formulating mathematical questions was by students looked upon as a strange task. By habit, the very restricted task to *answer* a specific mathematical question was preferred, not the unrestricted task to find questions. But this seemed to be only an initial problem. Students also have reported learning from other students' communications.

When references were investigated for the project, it proved to be virtually impossible to find previous projects where the course material was intended to be modified as a result on the student feedback. Three projects were found, Frith, V. Jaftha J. & Prince R. (2004), Larson, T.R. (1999) and Porter G. J. (1995). In none of them a text book was under change – all referred to web material.

It was equally difficult to find such projects for elementary school or high school. Of course, teachers learn from dialogues with students, and text book authors attempt to reach real students. However, in the absence of systematic ways of doing this, the image of students' mathematical problems may mainly be formed by those students that teachers talk to, while other students have different unformulated problems. An author often writes the text to fit an ideal student. How well does this ideal student correspond to real students? The answer to this question is of course of basic significance for the value of the text book.

A main aim of the project was to make the image of the ideal mathematics student more realistic, both in requiring feedback from all students, and by letting the students make the formulations by themselves – not directed by teachers' questions. One inference of the lack of similar projects is that the teacher culture does not value the importance of systematic student feedback for long term improvement of education – other than the natural feedback that takes place in mathematics classrooms.

In the next section we argue that the formulation of a systematic image of students' view of mathematics, available for teachers, may be of fundamental importance for the quality of teaching.

Reflective mathematics – defragmentizing mathematics

The purpose to discuss the nature of mathematics is here to *reach more reasonable teacher expectations* towards mathematics and mathematical activity, particularly in any kind of mathematics education. The purpose is to avoid teaching practices that fail, and where the reasons for the failure become clear years later. It is to be more prepared for events in the mathematics classroom, and to be able to design successful didactical projects. This is of course a fundamental purpose of mathematics education in general.

Avoiding failure requires many kinds of insights and competencies, but we here focus knowledge in "reflective mathematics", which concerns meanings of mathematical concepts and calculations in formulations accessible for students. It is possible to do very good mathematics without ever being aware of this mathematical knowledge – without the need to formulate it. This is a common circumstance in linguistics in the sense that we constantly may improve in our native language without the need of being grammar-conscious. This is relevant for mathematics in view of the dominant symbolic language. An underlying assumption here is that symbolic mathematics poses the main problems for the student collective, and reflective mathematics is intended as a bridge to symbolic mathematics. With this purpose, the two need to be tightly connected.

Aiming at the concepts of mathematics "underlying" formulas, we start by discussing the meaning of "conceptual" in mathematics. In the context of the meaning of "understanding", Anna Sierpinska describes that "*The distinction between "seeing" and "seeing as" is important for mathematics whose very nature does not allow for "seeing" its objects, but always to "see them as*", see Sierpinska (1994), p. 10. Thus, conceptual descriptions in mathematics are in principle always metaphoric. About "conceptual representation" and "conception", Sierpinska writes: "*While a conceptual representation is defined as expressible totally in words, a "conception" may be very intuitive, partly visual and not necessarily logically consistent or complete. A person who has a "conception" of, for example, the mathematical concept of a limit, "has some notion" of it, has "some understanding" of it not necessarily of the most elaborate level."*

For Sierpinska, a "conception" does not have to be expressible in words. In both cases it concerns mathematics understanding that is not restricted to symbolic representation. The relation of symbolic versus non-symbolic representation of a concept will sooner or later be important. However, it is important to be able to communicate and elaborate concepts before the "symbolic state", which thus need to be made in native language, images and other ways of expression. Reflective mathematics cannot restrict itself to symbolic representation, but should be related to it.

As described above, school work is a major source for reflective mathematics. All cultures that are involved with mathematics may contribute. The term "mathematics" has very different meanings for mathematicians, elementary school mathematics teachers, journalists, technicians, students, parents. A conceptual discussion is needed in order to start to understand these different views. We will later discuss the dialogue seminar, which is a natural tool for such cross-cultural interchange.

Conceptualities of mathematics are often discovered by teachers during their practice. Here teachers are forced into discovery by the pressure of students in need and engagement by teachers. Repeated such discoveries and explanations are extremely valuable for text book teachers and the mathematics culture. Such discoveries may require reformulation of fundamental mathematical issues. They are important since their source is students' work.

We give next an example to further describe the notion of reflective mathematics. It is a result from the author's dialogues with students.

An example of reflective mathematics

As an example of reflective mathematics we describe the two major mathematical generalizations that children encounter in elementary school. The first goes from sets of identical objects to numbers – which represent the cardinality of a set – the number of objects of the set. The second goes from numbers to letters – which represent numbers. The first connects reality to mathematics, while the second generalization is inside mathematics, since both numbers and letters belong to the mathematics realm. Both represent conceptual difficulties for children, which cannot be expected to be overcome by calculation practice only.

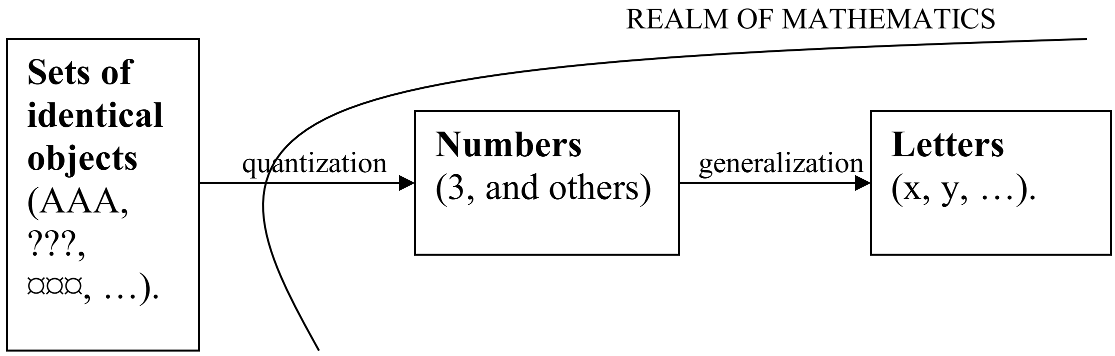

Figure 1 Two mathematics generalizations in elementary school: from sets to numbers, and from numbers to letters.

The first generalization provide children with appropriate meaning to the number symbols 1, 2, 3, …, hopefully. However, teachers working with children with special difficulties rapidly recognize the depths of abstraction that are embedded in these extremely common symbols, irrelevant and unseen for others. Most of us learn to calculate and use calculations in our everyday life, which does not necessarily mean that we "understand" (nor need to "understand") these symbols (Sierpinska, 1994, Smith 1994). This is well in accordance with the linguistics of mathematics. We may well learn and do well at a surface level, without even being aware of the existence of deeper levels. This is of course not always true, as for example the second generalization indicates.

This is underlined by a work by Skemp (1982), who identified two levels where students may work: surface/syntactic level and deep/semantic level. Some students may try to master the "symbol practice" itself, while some may try to understand and work with the underlying meanings of the symbols. Goodchild (1997) found from empirical material that almost all students follow either one of these ways of work. They were reluctant to switch level, either to make sense of syntactic operation, or to facilitate complex tasks by using an efficient formalism.

Before the second generalization, children among other things learn operations from addition to division, calculation methods, the place value system, the decimal point, and more. Despite the quantity of number manipulation in school, numbers are among children not often considered as objects that may be combined and manipulated. Again from Sierpinska (1994 p. 7), who discusses Greeno (1991): "*It is a very poor understanding, Greeno says, if a person, asked to calculate mentally "25·48" represents to himself or herself the paper and pencil algorithm and tries to do it in his or her head. A better understanding occurs if the person treats 25 and 48 as objects that can be "combined" and "decomposed": 48 is 40 and 8…"*

In Andersson, A-F & Bengtsson, F. (2001) two teacher students describe how they compared the mathematics knowledge in two fifth grade classes, where one class had little mathematical dialogue, and one had much dialogue. Of course, many other factors varied, which may influence the findings. The first class was slightly better in calculations than the second. However, when given the question "In which ways can you write 5?", children in the first class typically did not answer at all, while children in the other class filled the paper with calculations as "1 + 4, 2 + 3, 6 – 1, 25/5…". For them, the number 5 was obviously decomposable. The students in the second class had certainly seen this type of question before, but the point here is that students may have very different views of the flexibility and decomposability of numbers.

To summarize this discussion, one could say that it would be a large conceptual gain in mathematics understanding if all children regard *numbers as objects that evidently can be combined and decomposed in many different ways*. Children may see a similarity between numbers and construction toys such as Lego, for example.

This reflective mathematical observation can be extended slightly. Sometimes the converse question appears, for example whether 1/2 and 0.5 is the same number. A strongly related statement is that all numbers can be written in many ways. Does the question whether 1/2 = 0.5 or not arise from a misinterpreted uniqueness of mathematics, saying that "symbols which are different have different meanings"? Such a notion could be counteracted by establishing the obvious existence of synonyms in the formal language of mathematics, as well as in Swedish, English and other natural languages. The metaphor of "number line" for numbers can also help, in that 1/2 and 0.5 are represented by the same point on this line.

The fact that *any number can be written in many different ways* is fundamental for mathematics, since the main part of most mathematical proofs consist of rewriting the same expression in such a way that is more suitable for the goal of the proof. Without this synonymic property of mathematics, one can therefore question the possibility of mathematical proof.

Note that these two views of numbers, as combinable and decomposable on one hand, and as naturally having many synonyms on the other, is mathematical knowledge that is not often well established in text books. Furthermore, these statements cannot be written in symbolic language. Such observations do not appear by themselves from hours and hours of calculation. Some kind of appropriate mathematics reflection is needed. These statements are examples of reflective mathematics.

We also shortly comment the second generalization during elementary school, from numbers to letters. It is easy for teachers, but may appear very strange for students. Teachers often say that one can do the same thing with letters as with numbers, and we think of the fact that letters may be replaced by numbers, so the same rules are valid. But in many other obvious respects, which students may have in mind, this is not true. For example, it is not possible or meaningful to transform the number "x" into decimal form, as can be made with 1/4. Furthermore, the goals of calculation are entirely different. It is possible to calculate 2 + 3 and end up with 5, or calculate 345•73 and use a certain way of structuring the multiplication. Nothing of this is relevant when numbers are replaced with letters. We may consider that x + y = y + x, but do not calculate anything. We contemplate, summarize and discuss rules of calculation. Students get tasks such as to simplify (1/x + 1/y)/(1/x – 1/y), although it may not at all be clear when this goal is reached. Actually, such goals cannot be strictly specified. One strict way would be to count the number of symbols in the answer, but this does not always give the "mostly simplified" answer according to the mathematics culture. This difficulty is related to the famous assertion by Wittgenstein that there are no rules for how to use rules.

So a teacher who claims that one can do the same thing with letters as with numbers refers to formal truth, but not to activities and goals. The exceptional focus on truth itself can also be observed in research reports in mathematics, where the main question is the truth of results and why they are true, i.e. proofs, while the result's meaning, significance and relevance usually receive minor attention. Reflective mathematics tries to formulate this second aspect, which obviously is essential for students' mathematics learning.

The subject of mathematics drastically changes its character at this generalization, which a teacher who focuses formal truth may not notice. Different mental capabilities of the students become important. This change is known to often cause problems, which should be taken into careful consideration in text books and by teachers. Dick Tahta formulated a classical dilemma in (FLM 1984), as one of the two most obstinate longstanding problems: *Why is traditional algebra so difficult for a large majority of students?*

Perhaps the answer to this question is absence of reflective mathematics, and thus dialogue, since reflective mathematics is a result from dialogue. It would be a defeat for human communication if experienced teachers simply cannot adequately understand students' problems with algebra even after repeated and sincere student-teacher communication with no time limitations. The outcome of such communication is by definition reflective mathematics.

Thus, "reflective mathematics" aims at being a general and metaphorical description of formal mathematics, providing more meaning and overview to formulas and concepts, essentially making formal mathematics more accessible. However, the development of reflective mathematics relies on the courage of any mathematics active person to try to formulate significant mathematical problems, questions and considerations from ones own authentic personal viewpoint. Reflective mathematics can grow from mathematical dialogue concerning real questions, including those that occur between different mathematics cultures. We have no stronger tool for thinking than our native language. We cannot do without this tool if we want to formulate central conceptual facts in mathematics, regardless of the shadows cast by its formidably powerful symbolic language. This language is powerful but can only express a part of the essential mathematics knowledge.

Formulation of reflective mathematics requires a fundamental change in attitude to mathematics, towards an attitude that is more akin to that in the humanities. Mathematical errors are not only disturbances to be corrected, but potential sources of discovery of reflective mathematics, and opportunities for respectful dialogue. Each person's pronounced view of mathematics is important in its own, and may be important for the formation of reflective mathematics. The concepts of mathematics can be defined as the meaning of its formulas, and they are both abstract and not easily described in writing. Typically, they need dialogue to come alive.

The dialogue seminar

The scientific development in mathematics since the birth of the symbolic notation has been very fast. It has often been developed formally only, with meaning sometimes arriving later. Both the speed and the formality can be related to the power of this language. Certainly, an underlying idea of the formalist approach by Hilbert and mathematics logic project of Frege was that formalism is self-sufficient. As described above, conceptual observations in mathematics, are preferably made in dialogue, also between different mathematics cultures (teachers, students, mathematicians), which represent different mathematical experiences.

The dialogue seminar is partly designed as a framework for cross-cultural dialogues. We do not here give a full description of the method, but in Göranzon & Hammarén (year???), major goals are described as follows: "*The dialogue* *seminar method is a method of working that aims to (i) create a practice of reflection (ii) formulate problems from the dilemma (iii) work up common language (iv) train the ability to listen*." Furthermore, "*As a method, the dialogue seminar expands the perspective of the concept of knowledge by extending its field to encompass the nature of practical knowledge*." The dialogue seminar does not only aim at knowledge that can be written. Also practical knowledge and skill are central. This is well in accordance with the teaching profession that clearly does not rely on knowledge only – there is also a large component of unformulated skill.

All these four goals are important for the development of reflective mathematics: (i) create reflection on mathematical activity, (ii) viewing difficulties ("dilemmas") as opportunities of better understanding of mathematics and how it naturally is understood, (iii) to create a nuanced language about mathematics that complement the formal language, and (iv) to train different mathematical cultures, mainly students, teachers, mathematicians and teacher educators, to listen seriously to each other.

The dialogue seminar requires collective work to continue over time. It works with examples, both from the involved individual's experiences, and from the literature. The participants are "coordinated" by studying one common text. Each participant actively prepares herself/himself before the seminar by writing a reaction on that text, possibly from experience. During the meeting each participant reads the text for the others, after which comments are allowed, while criticism is not.

Furthermore from Göranzon & Hammarén (???)_ *In Plato's writing on Socrates' dialogue, dialogue is an instrument of understanding. But the understanding is of a special type, and is never a synthesis. It is based on a concept of truth that can never be captured or made permanent*. In this view, text books do not contain knowledge of this type. Text books contain mere images or shadows of knowledge, from which knowledge may emerge under benevolent circumstances. This poses two tasks to text books: 1. containing a selection of the most appropriate "shadows of knowledge", and 2. to communicate that this knowledge is only "shadow knowledge", and to suggest developments.

This observation also indicates that in Plato's view, practical knowledge is a knowledge that is essentially too complex to be written, but can be made visible in a group of listening, engaged and experienced persons. Plato's note also indicates the dangers in languages. It is tempting to see linguistic expressions themselves as knowledge.

In Sfard (2005) the author states that "*The teacher could hardly be blamed for being a captive of her own dicursive ways. While in the midst of intensive interaction with a group of children she could not allow herself the luxury of multiple interpretations.*" Sfard claims that reflection on practice is difficult from the inside, it needs an outside view. She furthermore describes the possible power of educational research: "*The power of educational research lies in its being the art of multiple interpretations. By making clear that there are many narratives to be told about any given instance of educational practice, this research loosens the oppressive grip of old discursive habits and sets us free to consider new options."* This view of educational research is very much in parallel to the goals of the dialogue seminar.

In Järfälla outside Stockholm, Sweden, the project "Höja nivån", led by Pi Högdahl, has significantly decreased the number of students that leave elementary school without a grade in mathematics – see Högdahl (2005). This result has been achieved by providing mathematics teachers time and opportunity to meet and discuss mathematics and educational problems from the practice in their mathematics classes. Continuing this, a dialogue seminar has recently started in these schools, supported by the Swedish National Agency for School Improvement. It is led by Pi Högdahl, Håkan Lennerstad and Martin Gode, and has as central theme translations between mathematics formulae (Mathematish) to Swedish. This has the purpose of, for students, charging strange mathematical formulae with concrete or dramatic meanings, demonstrating the rules of formulae in detail, and encourage natural language in mathematics class. It attempts to shed light upon the linguistic difficulties in mathematics that appear in practice. Teachers meet and reflect about such translations and their value in practice.

In Ericsson Å., & Söderström C. (2006) the outcome of the dialogue seminars during the fall of 2005 are documented. Teachers were in general very content with this form of professional development, allowing rich opportunities to express and listen to teacher experiences. Teachers developed also understanding of linguistic properties of the symbolic language of mathematics – Mathematish. For example, Mathematish synonyms were often talked about.

There does not seem to be either very difficult or very time consuming to start to complete mathematics with reflective mathematics, and at the same time develop teachers' skill by teachers' organized exchange of experience.

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