An Evolutionary Text Book – Evolving by Student Activity

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Abstract

This paper presents a project where a calculus text book was updated by using student dialogues that occurred at a web page. The web page invited student activity and influence on the text book, and the dialogues often concerned specific solutions and arguments in the book. This project is described together with issues about different types of knowledge that are important for teachers, and that were raised by the project.
To characterize common learning obstacles for students in a particular subject, authentic teacher-student dialogues during a course are an essential material. It is extremely valuable for text book authors and for teachers, but they need reflection to be usable. If a text book repeatedly, year after year, is revised by authentic student activity, filtered by the author's competence, the book may evolve to be well adapted both to student's expectations, formed by previous schools and the present culture in general, and to the author's view of which kind of subject knowledge students should learn. Studies by students are of course the activity a text book is made for. It is reasonable that the book is regularly affected by this activity in order to become increasingly well fitted for the task.
In the project students were also required to formulate their own mathematical questions, not only to solve problems. The project has also implications in terms of students' sense of responsibility for their education, since their activity may change the education – making their education not only a personal project.
Empiric results are presented for an undergraduate distance course in calculus with 20 students.

1. Introduction

1.1. The project and types of knowledge

Teachers are communicators of subject knowledge to students. As such, it is important that teachers are familiar with student's existing knowledge and habits of thought in the subject. By several reasons, which we discuss in Section 2, teachers' knowledge of this kind needs to be constantly updated. This kind of knowledge is one of the two focuses of this report which is discussed in relation to other kinds of knowledge in Section 2. The paper also presents an internet based project allowing teachers' to develop this type of knowledge and to improve the text book. Background and plans for this project are described in Section 3. Results, evaluations, and possible generalizations are given in Section 4. In the remainder of this introduction, the paper, the project and its background are described briefly.

The report focuses university education, but is also relevant for elementary educations. It particularly concerns distance education, and educations where it would be desirable to increase the amount and quality of student-teacher dialogues. Mathematics is often described as a subject where dialogues are needed.

It is completely natural with a cultural divide between teachers and students. But this divide is a problem for the success of education if students understand the subject in ways that teachers do not recognize. Student's way of thinking is to some extent local, depending on particular teachers and school reforms in previous schools, on technology used, and on the view of the subject in society. It can only partially be covered in teacher education or in other education where students are not present. Therefore, teaching need to involve learning for both parts – both students gradually improving the understanding the subject, and teachers gradually improving the understanding of students' ways of understanding the subject.

Knowledge about the ways students think certainly develops for any serious teacher during practice. It is however very individual and depends on the teacher's attitude towards the teaching. It is an open question how large educational improvements may follow if this knowledge is deliberately cultivated, for example by systematic work among teachers. There may be a large difference between "spontaneuous" growth of knowledge of student thinking by years of teacher practice, and the counterpart that evolves if reflection with other teachers takes place, together with subsequent teaching modifications and renewed reflection. To inspire such systematic work appears to be precisely the aim of Huber & Hutchings (2005). They also point out the difficulties in evaluating such activity since the type of knowledge that is enhanced is qualitative in nature.

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Certainly, most teachers and researchers have the explicit goal of understanding students' knowledge. Paul Ramsden declares the goal of his book *Learning to Teach in Higher Education* as follows: "*The basic idea in this book is that we can improve our teaching by studying our students' learning – by listening to and learning from our students.*" Ramsden (2003 p.6). The present report describes a practical way to systematize this activity, to achieve such listening and learning, with the specific aim to improve the text book.

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This kind of teachers' learning takes place naturally during teaching practice to some extent, but it is very dependent on both the amount of dialogue and also on the amount of teacher reflection about the occurring dialogue. Since memories fade, absent dialogue may mean that teachers understanding of students' way to understand the subject deteriorate, and teacher-student misunderstandings increase. We never know how many teacher-student misunderstandings that loom in our courses. Some misunderstandings become visible and are resolved, but there may be severe misunderstandings that never reach the surface. Since mathematics often is a silent subject, particularly at university level, this may be a serious problem in mathematics. This is affirmed to some extent by the frequent reports about problems in mathematics learning (ref???).

1.2. In search for student oriented subject knowledge

The subject knowledge we look for in this paper can be defined like this: *subject knowledge that can be expressed in a text book on the subject so that misunderstandings are minimized and useful explanations are maximized, both from the perspective of present students*. Let us use the term *student oriented subject knowledge* for this kind of knowledge. This knowledge is clearly dependent on student cultures, and may be interwoven with images and other descriptions. The line between this kind of knowledge and its description is not distinct. The description is sometimes part of the knowledge – it may consist of valuable and effective descriptions.

In order to find and use this knowledge, the project *Student Influence on Text Books* was formed. It was funded by the Swedish Council for the Renewal of Higher Education and carried out at Blekinge Institute of Technology (BTH) during the period July 2003 – June 2004. The first author of this article, Håkan Lennerstad, was the project leader. He is also the author of the text book in question, which is briefly presented in Section 3 as a background of the project. The two co-authors of this paper, David Erman and Maria Salomonsson, participated as senior students in answering questions.

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The project involved two faculty members, including the project leader, five students answering questions and 39 students asking questions. In this report, the students and teachers responsible for answering questions on the website are referred to as *answerers*.

The course and the book cover calculus in one variable, since Sweden adopts the tradition of separating one variable calculus and several variable calculus in two separate courses. It is the first course in mathematics at the university for most students. It is covered during two or four months, and students normally study two other courses simultaneously.

1.3. Evolutionary text books

The main philosophy of the paper is that the most important source of text book improvement should be its natural use – students' learning processes when studying with it. *The book should change and evolve as a result of this process, for which it is made, to improve it as much as possible for future students' learning of the subject*. One may use a Darwinian metaphor. The environment in which a text book lives is students' working with it, and the book should be affected by the events in this environment and adapt to it in order to be increasingly better fitted for this use. Students' work is filtered by its creator – the author. Such work requires among other things the need for a channel of communication connecting students and authors – this is what the project provides. Such a channel requires students who are not afraid to formulate their questions and arguments, and authors who are able to listen, value such input, and are able to use it in a text book revision. Certainly, a text book needs also to be adapted to other environments to survive, such as the teacher culture and the market.

A text book is an evolutionary text book if the author breaks the isolation between the text and the experiences of the students who are using it – if systematic feedback from students to text via the author is possible.

As is described in Section 4, the outcome of the project was improvements of the book at particular points of mainly the following types: corrections, explanation of the non-generality of examples, logics of arguments, explanations with words and not only formulas, filling gaps in calculations, particularly handling of integration constants when solving differential equations.

This project concerned a text book that after the revision was printed in a new edition by a different publisher. An alternative is to have the entire book on the web as a set of PDF-files or as a Mathematica note book document, for example. This would be a different project that has both good and bad sides. If the book is entirely on the web, faster updating according to student comments is possible. This is however a minor point since many of the debate issues anyway are familiar for the students participating in the course. Economically it is preferable for students, which however decreases the possibility for financing projects. Many students prefer a printed book than the book electronically or as bundles of papers.

Is there a limit to the book revision process? Of course, if there are too few revisions, there is no need to print a new book or to force students not to be able to use the previous edition. It is up to the author and the publisher, considering pedagogical needs and economical constraints of teachers and students in schools to decide when the improvements are enough for a reprint. For some authors there may be a point where no further revisions are relevant, and the process ends. However, it is also possible that an author endlessly finds fundamental improvements as a result of student feedback. This depends on the author's view of the subject.

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1.4. Dialogue and feedback – some alternatives

The next question concerns how to achieve a valuable feedback. Teachers normally receive student feedback when grading exams and during dialogue during the course. These two types of feedback have drawbacks, which we discuss next.

During grading, students' ways of thinking is sometimes unclear for the grading teacher in a way that could be clarified by a dialogue. Also, students' arguments in an exam are strongly dependent on the exam problems. There may be many other questions about mathematics that are important for a student that never appear at an exam.

Dialogues during a course may suffer the risk of being dominated by a specific group of active students, while a more silent group may have unnoticed but very different views on mathematics. This bias is difficult to avoid given the limited time available for teachers.

In the project, a specific number of mathematical questions was compulsory for every student during the course. This guaranteed a sufficient material, but had also the effect of diminishing both these two drawbacks. Questions could refer to problems in certain calculations or refer to anything else that is connected to the course. Both silent and talkative students are expressing views, and problems that are not particularly exam-oriented are encouraged, and were in fact expressed. The students' identities were protected in that an alias could be used.

By obvious reasons, the quality of dialogue decreases when being on the internet and not in the real world. The communicative power of body language, images and symbolic language are all restricted. In the opposite direction point the facts that many students are very used to this kind of communication, and that the timing is more relaxed in email or forum communication than in verbal communication. Furthermore, it can be performed at home rather than at a university environment, which for some students may be less familiar and encouraging.

Thus, the purposes of the project were the following, here ordered by the emphasis given, with higher emphasis first:

1. To develop the readability of a text book from students' perspective.
2. To allow systematic improvement of teachers' familiarity with students' subject knowledge.
3. To invite students to a more responsible role in their education.

1.5. Evaluations and findings

We have made no attempt to measure the degree of improvement of the text book or the course, since it is extremely difficult to value such changes with any objectivity. The evaluations consist of verbal evaluations by the individuals that were involved.

The results would have benefited if also external evaluators had performed an investigation.
The findings of the project, as formulated in the evaluations, can be summarised in the following points:

1. Students were not used to posing questions in mathematics – they were strongly inclined to *answering* questions. However, they had little problem doing so after the initial difficulty. This led to a more open minded view of the subject.
2. The book has improved since several mathematical difficulties in the book that the students pointed out have been included.
3. Students in general appreciated the teacher feedback and the fact that their work and remarks could affect the book.
4. The project has the following overall result: *It is not difficult to arrange student-teacher dialogue in a distance course so that it is close to the existing text, discusses it, and thus inherently supports text book improvement. The most important requirement is that the author of the book values such a material. Text books used in distance courses may easily be evolutionary.*

2. Previous work and types of knowledge

2.1. Previous work

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Sometimes text books have been developed by students and teachers together, but there appear to be very unusual with a continuing systematic revision of a book after its publication.
The following related projects were all applied in undergraduate mathematics. In three of the projects, student feedback was essential for web based mathematics learning material. In the fourth, the effectiveness of corrective feedback in computer-directed instruction was investigated. Improving an existing text book was not the target in any of them.
We next present the first three of these projects in more detail.

In Larson (1999), the author summarizes the design and development of a participatory calculus textbook offered as a subscription site on the Internet. As an interactive multimedia textbook, it integrates text, graphics, animation, simulation. The text also features collaborative environments similar to the familiar online chat and news. The convergence of interactive multimedia course materials with context-sensitive collaborative environments is called a participatory document. The structure of this project represents over four years of design research while implementation represents several person-years. Hundreds of students have used the text and based partly on their feedback, additional projects are being prepared. The project "An Interactive Text for Linear Algebra", see Porter (1995), has supported the creation of a computer-based interactive text for linear algebra using guided discovery in a laboratory based course, emphasizing active learning, collaborative learning, and the use of writing. The article Frith et al. (2004) describes a study of learning with students using interactive spreadsheet-based computer tutorials in a mathematical literacy course. It foregrounds theories relating to the role of computer technology (and specifically spreadsheets) as a mediator for the learning of mathematics. There are indications that the data reveals real differences between the learning experiences in the lecture sessions and the computer laboratories. It appears that in some respects the computer tutorials has been more effective in conveying the concepts than the lecture session.

2.2. Types of knowledge that teachers need

Since there are few projects of this kind, one may ask if this reflects a gap in our view of which kind of knowledge that teachers need. Is student oriented subject knowledge important but neglected?

Many authors describe such kinds of knowledge. The term "epistemological obstacle" is coined by Bachelard (1983) and used extensively by Sierpinska (1994), refers to ways of understanding or misunderstanding based on cultural and other schemes of thought.

In *Mathematics Teaching Practice*, Mason (2002), the author attempts to formulate such epistemological obstacles in calculus. This is very much in line with the present project. These attempts have different degree of closeness to the subject. Bachelard and Sierpinska discuss the concept of "epistemological obstacle" partly by using examples, rather than attempting to find which such obstacles are common and important today. In Mason (2002) the author comes closer in describing actual obstacles, but directs himself towards teaching, not towards text books.

We think of student oriented subject knowledge as being formed mainly in student-teacher dialogue and student-student dialogue. This knowledge has a certain authenticity that may lose its value when it is reformulated by teachers and researchers. Teachers who are familiar with the subject may not recognize the importance of some metaphors valued by students. The formulation of this knowledge is important, and belongs to the students. An author who incorporates students' suggestions need to be careful so that the change still is useful for other reading students. Courses in pedagogical and didactical theory may provide an important basis on which to understand student oriented subject knowledge.

There are several arguments why student oriented subject knowledge can be learned mostly form teaching practice – from dialogue with students. The code word is "culture". The main theme is the difficulty for teachers and students to understand each other cultures enough to achieve a successful education. Cultures are not easily observed when viewed from the inside.

In Lennerstad (2006) it is discussed how such a student-to-teacher feedback could transform the teachers' description of mathematics towards one that has a larger accessibility for students. Here the so called dialogue seminar is proposed as a form for teachers to meet and formulate this knowledge. That paper touches upon a project where teachers in 9th grade met in dialogue seminars over the issue of translating between Swedish and the symbolic language of mathematics. This project was funded by the Swedish National Agency for School Improvement, and was reported in Eriksson & Söderström (2006). Teachers evaluated this as an opportunity to in depth share thoughts and experiences about students' mathematical thinking and to learn to listen.

2.3. Mathematics and communication

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It is well known that mathematics in general has a tradition of relatively weak teacher-student communication, particularly at higher levels. Silent work with the text book is predominant, more so than in other subjects. Accordingly, feedback from students to teachers is then limited. There are several possible reasons for this. The subject is abstract and far from the everyday experiences of students. Students often feel uncertain about the meaning of statements, concepts and use of symbolic language. In Lennerstad & Mouwitz (2004) the linguistic problems with the symbolic language are articulated – particularly how the presence of this language for students may limit the use of the student's native language. Many researchers today acknowledge that mathematics involves many tacit or implicit types of knowledge that are difficult to articulate, even for teachers and researchers, described for example in Ernest (1999).

3. Goals and background of the project

This project tries to give answers to the following questions, which are discussed and evaluated in Section 4:

1. What kinds of improvements of a text book can students contribute with?
2. Can a text book be significantly improved by student feedback?
3. Is it possible to extend this feedback model to other subjects?
4. Does this project provide other benefits for students' learning?
5. How much extra resources in time, personnel and equipment are necessary?

3.1. Prerequisites

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This project assumes certain obvious fundamental practical requirements such as a course where the text book is used and a suitable course web site where dialogues conveniently can take place. Funding is welcome for teachers and advanced students who answer questions, but is not necessary if the students themselves are encouraged to answer questions.

A further fundamental prerequisite is the attitude and pedagogical knowledge of the author, in order to be able to use the student material for text book improvement. The project requires an active interest in the students' way of working as well as a mathematical pedagogical competence of how their questions can be transformed into text book improvements. For example, a proof can be written in different ways, such as starting by the assumption or starting by the result. It may be preceded by a typical example where the calculation is very similar to that of the proof. Also, many proofs can be described geometrically. There are many dimensions available for the author both mathematically and for the sake of simplifying students' understanding.

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3.2. The text book used

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This section briefly presents the text book which was used in the project, and which covers the course content. The text book "Envariabelanalys med dialoger" (English: "Calculus with dialogues") consists of 774 pages. It has the calculus content representative of Swedish standards. However, the large number of pages is due to appearing student dialogues to make it easier for students to identify themselves and at the same time put forward central concepts. Furthermore, the book contains many examples with full and detailed solutions.

The book is not a very typical Swedish text book. However, we next give a translation of its table of contents – of its chapters. The 17 chapters are collected in four parts. The first three chapters are somewhat unusual, so for these we include a short description.

Part I – Numbers and functions
1. **What is university mathematics?** Discusses mathematical branches, aims, goals, methods, graphics and the summation symbol. For example, functions are described as being at the heart of calculus since its main problems concern either properties of functions or equations with functions.
2. **Numbers and calculations.** About the five types of numbers, from integers to complex numbers, and which calculations are possible with each type.
3. **The function concept.** Introduces functions formally, geometrically and intuittively, both with abstract concepts and very simple examples. Main issues for the sequel are composition of functions and inverse functions. The main function classes are presented in the following three chapters.
4. **Polynomials and rational functions**
5. **Power functions, exponentials and logarithms**
6. **Trigonometric functions**

Part II – Limits and derivatives
7. **Limits and continuity**
8. **Derivatives**
9. **Derivatives – mathematical modelling**
10. **A little about McLaruinseries**

Part III – Integrals
11. **The integral concept**
12. **Indefinite integrals and primitive functions**
13. **Definite integrals and generalized integrals**
14. **Integrals – mathematical modelling**

Part IV – Differential equations
15. **General and separable differential equations**
16. **Linear differential equations**
17. **Differential equations – mathematical modelling**

3.3. Software functionality

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The URL for the web site is www.bth.se/analysmeddialoger. The web site is constructed, using PHP with a MySQL database backend. There is one entry on the web site for each page of the book. The intention is that a student will be able to see previous dialogues about events on a certain page in the book, and take advantage of this.

It is possible to post questions anonymously. Each questioner chooses an alias and password. A password is only required for the purpose of posting messages on the web site, not for browsing the site. The subjects of the messages are displayed in a tree structure, where related messages are in the same branch. By clicking a subject, the full message is displayed. When browsing an answer, the previous question is also visible.
We next include a short list of translations of Swedish word that occur on the home page into English.

Swedish	English	Artiklar	Articles
		Butik	Shop
Analys	Calculus	Logga ut	Log out
Matematikkarta	Mathematical	Debattinlägg	Debate post
map		Alla inlägg	All posts

Om boken	About the book	Uppg.	Task
Tyck till	Express a view	Replikera	Replicate
Registration	Registration	Status	Status
		Alias	Alias
Gå till	Go to	Skapad	Created
Nästa	Next	Öppna	Open
Inlägg	Post, contribution	Besvarad	Answered
		Svar	Answer

Rather important for the flow of communication are the following email features. When a question has been posted, an email is sent to each answerer. After the question has been answered, an email is sent to the questioner. The emails contain a link to the web site with the question and the new answer.

Mathematical formulas are expressed with normal fonts, which do not appear to have caused any problems for the people involved. A little imagination is required, such as for example to understand an expression such as “integral(from 0 to 2) x^2/(x^2 + 1) dx”.

One may argue that this is confusing in that students need to learn a new notation system. It is also possible to argue that this simplified notation provides a detailed explanation of the standard notation, and hence to some extent explaining it. There has been no complains to the notation system. Nevertheless, new tools develop, so future projects may perhaps easily use full notation.

3.4. The mathematical map

The inner sleave of the book contains a so called mathematical map. It gives a metaphorical overview over the entire mathematical content in the book. A mathematical map may contain mountains, forests, rivers, lakes, cities, countries and so on, and all names on the map are mathematical words or formulas. The most important mathematical concepts in the course are dominating features, such as large cities. Different countries may be sub-contexts, perhaps chapters, where items in different contexts are connected by rivers, forests and mountain ranges. There are large possibilities to use geographical images to represent mathematical connections or meanings. The sequence of pages in the book can be said to correspond to a certain walk in this landscape.

We next describe briefly how this particular map describes the calculus content in this course, sketched above in the table of content. The map is divided in eight countries: Numbers, Function, Sets, Limit, Continuity, Derivative, Integral and Differential Equation. The smallest country is Sets, and the largest is Function. The country Function consists in turn of seven provinces representing the main classes of elementary functions. In each region main items are represented, such as the Fifth Degree Mountains for the difficulty of solving the general fifth degree polynomial equation explicitly. Function also has areas outside these provinces, where general properties are represented. Differential Equation is divided in Linear and Non-Linear. At the border between these two parts is the city Separable, since there are both linear and non-linear separable equations.

The map is dominated by two large rivers: Equation Solving in the west and Composition in the east. Equation Solving has its source in Numbers in the north, and passes by the villages **N** (natural numbers), **Z** (integers), **Q** (rational numbers), **R** (real numbers) and **C** (complex numbers), since the five types of numbers have arisen when attempting to solve polynomial equations. For example, $x^2 = 2$ promted real numbers, and study of equations like $x^2 = -1$ gave rise to the complex numbers. The river Equation Solving is important also in Differential Equations in the south. This river does not pass through Integral, since in this context equations are not studied within this calculus course.

The river Composition bifurcates from Equation Solving at the city Inverse, since inverse functions concerns both equation solving and composition. The functions cos*x* and sin*x* have their inverses arccos*x* and arcsin*x* at opposite sides of this river, as other inverse pairs. The river passes the cities Chain Rule (differentiation of a composite function), Variable substitution (integration of a composite function) and Separable Differential Equation (an equation where the solution is composite). The proofs of the last to are only applications of the chain rule – they are merely reformulations in different contexts. Actually, from this connection by composition, the idea of the map sprung, when composition as a red thread between major items was replaced by a blue river. The river metaphor allowed the growth of a landscape also away from the river, which not is suggested by a red thread. This expands the metaphor.

Fig. 1. The central part of the first mathematical map in English.

The included part of the only mathematical map so far in English, Fig. 1, depicts the central small country Limit with its volcano ∞, and the two villages ε and δ at its foot. The River of Calculations leads through many curves and bends over the Plains of Standard Limits, out of Limit, into the Plains of Standard Derivatives in Differentiation. After passing through the large city Fundamental theorem of Calculus, the same river traverses the Plains of Standard Integrals, now under the name Primitive. There is also the northbound river Approx, passing Asympthote, Normal and Tangent. The red dots mark walks on the map defined by the narrative of the book – its word by word linear order. Numbered squares in a circle or a square are starting and ending points, respectively, for the chapter with that number. The full map can soon be found at mathematics.lennerstad.se.

The map in the book, published 2002, is the first mathematical map. Since then the basic idea has been used as a pedagogical tool. Student groups have designed their own maps about mathematics. During mathematical map making, pupils discuss connections between the mathematical concepts and calculations they know, and produce an overall picture. Mathematical maps have been made at all levels, from kindergarten to university. The geographical metaphor turns out to be very powerful and central for the activity. It is both extremely well known and flexible, students find many innovative similarities between the geographical and mathematical domains, expressing abstract mathematical thought. In this type of mathematics learning, numbers and quantity are not the objects of thought – it is mathematical ideas themselves. Instead of correct answers, focus is upon the overall picture of mathematics that they know. Mathematics teachers have reported that they learn about students' mathematical reasoning by listening to ongoing dialogues during map making. Groups of pupils have rapidly grasped the idea and started to discuss, design and draw their maps of (their) mathematics. See Lennerstad (2007) for a thorough discussion of mathematical maps.

3.5. *Integration in courses*

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In some courses, web site activity has been voluntary and the course has not been given by the project leader. In general, students have not used the web site in these courses. The teachers have later described the situation as the students already having enough material and opportunities. The web site has not been necessary for them, and furthermore students are not used to writing questions. The web site was not sufficiently integrated in these courses. The need for integration has been pointed out by the visiting Council representatives. Following these comments, changes to the use of the web site have been made. The students have first been rewarded by extra exam credit for asking and answering questions in late 2003, and in the distance course in early 2004, questions were made compulsory.

3.6. *Integration at the department*

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A practical issue regarding the project has been that of the project leader not meeting mathematics teachers on a daily basis. Mathematic teachers work in the town of Karlskrona, Sweden, whereas the project leader has his office in a computer science group, for research reasons, in the town of Ronneby, Sweden. The project would also have benefited from a more thorough introduction by mathematics teachers at BIT, encouraging the use of the web site.

4. *Results and generalizations*

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4.1. *Presentation of the material of dialogues*

Next a characterization of different types of notes on the web site is presented. In total 29 students contributed and 7 answerers were active. The notes and comments on the site are categorized into four categories: Questions on problem solving, questions on ideas, words and concepts, questions on applications and history and general feedback to the author.

The following table details the questions and the distribution of questions according to category, together with an indication of what amount of comments resulted in changes of the textbook. The last two columns indicate the number of comments, and the number of comments that resulted in a direct change ("yes") or not ("no"). In total, 569 comments or questions were placed on the web site.

Question type	yes	No
Questions on problem solving		
Understanding the problem situation. What is being asked?		13
Full calculation - wrong answer. Typically a long calculation but the wrong answer, and the question: what is wrong?		50
Full calculation - check of correctness. Similar to the previous question, but with a correct answer.		9
Little calculation and unclear question. Diffuse questions on how a certain problem could be solved.		32
Clear method question. A question where different methods are mentioned and compared.	42	123
Questions on ideas, words and concepts		
Questions on concepts and ideas. Questions on prime numbers, open and closed intervals, dense set, inverse function, meaning of mathematics, relevance of figures	49	101
Theoretical remark. Discussion about a theorem. Sometimes a further explanation is added.	4	6
Numerical aspects.		3
Geometrical interpretation.		8
Notations and words. *Example*: Why is limit denoted by "lim"?	5	44
Questions on applications and history		
Questions on applications. *Examples*: In the limit $20(1 - e^{-t})$ when $t \to \infty$, which may be the temperature in a house, do we never reach the temperature 20°?		6
Mathematical historical questions.		13
Comparison with other book.		5
Feedback to author		
Suggestion of improvement.	5	4
Suggestion of correction.	27	
Question about graphs.		10
Appreciation of explanation in book.		7
Superlatives on beauty or insight. A few parametric curves are in the book just for their aesthetic value, $(\cos t + e^{\sin t}, e^{\cos t})$ and $(\cos t + e^{\cos t}, e^{\sin t})$, which resembled rocks at a shore. These have triggered the comment "Mathematics is beautiful! And not only in an abstract way to be appreciated by the professional mathematician!"		3

4.2. Evaluations

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This section presents evaluations by students, answerers and the project leader.

4.2.1. Evaluations by students

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To describe the evaluation, we first shortly describe the examination in the distance course. It has consisted of four written assignments of examination character where group work has been encouraged, (at least) 42 compulsory mathematical questions on the web page and an individual examination at the end of the course.

Students have used two web sites in the course: one containing course schedule and all other information. The other has been the web site of the book for mathematical dialogues. 32 students have been registered, and 11 have passed the course. Recorded video lectures about the main issues have been available on the course web site. In these recorded lectures a student sees pages from the book and a pointer, while the audio is the author commenting and explaining what is written. Comments complement the written text in that the main points are emphasized and that a less formal language is used.

The enquiry is divided in three parts: A. The course, B. The book, C. Questioning and the web site. Twelve students have participated in this enquiry, but not all questions have been answered. For each question there has been a choice between five degrees of evaluation. The columns contain the number of students giving a particular answer.

A. The course

Query	Reply range	5	4	3	2	1
The course content appears to be?	Very relevant – not relevant	5	5	1	0	0
Course difficulty level?	Easy – very difficult	0	3	3	4	0
The amount of material is?	Too little – too much	0	1	9	2	0
A comprehensible course plan?	Very – not at all	1	6	3	1	0
The course staff replied rapidly.	Very good – not good at all	4	6	2	0	0
The technical parts worked:	Very good – not good at all	4	6	1	0	0
The course gave me valuable knowledge of mathematics.	Very much – not at all	1	8	2	0	0
The course made it easier for me to solve mathematical problems and work with mathematics.	Yes – no	1	6	4	0	0
The course made it easier for me to see and understand mathematical connections and concepts.	Yes – no	2	6	2	0	0
The course made it easier for me to see connections between mathematics and applications.	Yes – no	0	4	6	2	0
The course was compatible with previous courses within the program.	Yes – no	0	7	2	1	0
I used the recorded lectures on the web.	Very often – never	3	2	6	5	0
From the point of view of problem solving, the recorded lectures were:	Very useful – not useful at all	1	3	5	3	1
In order to understand mathematical connections and concepts, the recorded lectures were:	Very useful – not useful at all	1	4	5	2	1

B. The book

Query	Reply range	5	4	3	2	1
Readability. The book is:	Very clear and easy to understand – very unclear and difficult to under-	2	8	1	0	0

Query	Reply range	5	4	3	2	1
	stand					
Knowledge (text + figures, etc.). The book makes it:	Very easy to understand mathematics – not at all easy to understand mathematics	3	4	4	0	0
Practice of problem solving. The book makes it:	Very easy to learn problem solving – not at all easy to learn problem solving	0	6	4	0	0
The book has improved my ability to use mathematical formulas.	Very much – not at all	0	10	0	0	0
Overview. The book gives an overview of the mathematics that is:	Very good – not good at all	0	3	7	1	0
Priority. The book emphasizes what is most important	Too much – too little	0	3	7	1	0
Long term knowledge. The book explains so that mathematics I meet in the future will be	Much more easy – not at all more easy	1	9	1	0	0
Level. The book appears to be	Very mathematically advanced – not at all mathematically advanced	0	7	4	0	0
Create interest. The book makes mathematics and mathematical problems to be	Much more interesting – very much more boring	3	6	2	0	0
Attitude. To look in the book is	Actually fun – not at all	0	8	2	1	0
Working time. I have been working with the book	1. More than 20 hours each week 2. 10-20 hours each week 3. 5-10 hours each week 4. 1-5 hours each week 5. Less than 1 hour each week	1	2	7	0	1

C. Questioning and the web site

Query	Reply range	5	4	3	2	1
In order to learn mathematics, I think that to regularly write mathematical questions on a web site is:	Very good – confusing	0	4	4	1	1
Having compulsory questions is:	OK – not OK at all	0	0	5	3	2
The answers to my questions have arrived:	Rapidly and always – slowly or never	1	5	3	1	0
The answers to my questions have been:	Very good – confusing	2	6	1	0	0
From the answers I have learned:	Very much – nothing	0	8	2	0	0

Query	Reply range	5	4	3	2	1
I have been reading other questions and answers at the web site.	Often – never	0	2	4	4	0
From questions and answers other than mine I have learned:	Very much – nothing	0	2	3	4	0
To give questions on the web site has changed my attitude to mathematics.	Very much positively – very much negatively	0	2	8	0	0
On-line-help. A web site where you can give questions on the book and receive answers is:	A very good idea – not useful	5	5	0	0	0
Student influence. That student problems affect future versions of the book is:	A very good idea – not useful	5	4	1	0	0
The fact that my questions may affect the next version of the book has for my learning been:	Very positive – very negative	2	3	5	0	0

4.2.2. Evaluation by answerers on their mathematics learning

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The comment by David Erman and Maria Salomonsson here follows with the focus on the mathematics learning for themselves as participants.

"As senior students, the opportunity of being involved in the answering of student queries has been very beneficial in several ways. Firstly and foremost, it forces us to write as lucidly and explanatory as our ability allows, and naturally hones our skills as conveyors of information. Secondly, it is a humbling experience, as we are reminded of our own fallibility and lack of knowledge in the areas covered by the book. Several questions posed by students have forced us to return to our own notes and books to find answers for *ourselves*, just to be able to give satisfactory explanations to the query. Thirdly, we gain a fair amount of new views of the mathematical concepts by reading the questions. This of course gives a wider and more complete grasp of the topics covered. Fourthly, many of the student questions have been regarding the history and stories behind the mathematical notations and people. It is highly unlikely that this information would have been sought out independently by us without the incentive of the CwD site. Not that the interest isn't there, but rather that once you consider yourself "done" learning a topic, you stop wondering. Mathematics has been made a living entity again, in a sense."

4.2.3. Evaluation by the project leader on the project flow

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The project has suffered from the fact that the project leader has not met the other mathematics teachers on a daily basis, due to research activity elsewhere. If this contact would have been closer, the teachers would probably have been more familiar with the book and the project, and would then naturally support the use of the web site.

4.3. *Evaluation by the project leader of the book revisions*

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The quantitative account of questions in Section 4.1 is here completed by a qualitative one, directed towards the quality of book revisions which have resulted from the questions. The qualitative evaluations are made not in terms of future students learning, for natural reasons. They are made in terms of what the present students wish, and the author's view of the value of the communications for the revisions of text book.

4.3.1. *Improved understanding of students' knowledge*

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This section contains some mathematical issues which have received large attention from students, and the resulting revisions of the text book.
Generality: One may claim that the most fundamental character of mathematics is its generality. This is reflected in student questions, since a very common question in all chapters has been: "Is this general?" Many revisions of the book say: "This is only an example, illustrating a concept". Formulas are extremely compact, though general. Their applicability and scope are never obvious.
Non-formal descriptions: Many revisions have further included non-formal descriptions connecting to the opinions of the students. This has been limited, however, since the book was already written in this way.
Further explanations of calculations: Calculations are often abundantly commented in the text book. However, more comments on particular steps have been added by student questions. Such changes are not necessarily improvements, since the overview over the calculation may decrease.
Constants appearing in integration: Many questions concern the constants that appear in integration, including what the expression "they differ at most in an additive constant" means. Revisions are added that connect the undetermined constant in integration to the fact that a derivative only concerns the change, independently at which level (height) from which the change occurs.
Constant appearing in solving differential equations: Many questions also concern the constants that appear in integration. In separable equations, usually two constants appear from x-integration and y-integration, which may be merged into one constant. Furthermore, in solving linear constant coefficient second order differential equations, two kinds of unknown coefficients often appear – in the homogenous solution and in attempting to find a particular solution. Further explanations are added to clarify this.

4.3.2. *Text book improvements*

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The following is a list of types of revisions of the book, sorted in order of value for improving the book, with the most important revision first, rated by the author.

1. Errors in the book have been found and corrected by student activity.
2. Explanations have been added on fundamental ways about the way mathematics work, what mathematical results attempt to say and the meaning of concepts. This concern the issues described in "Generality" and "Non-formal descriptions" above. Such explanations are very basic for students. It provides an overview of the ideas, putting detail problems in a more relevant perspective. In the absence of overview, many students worry about detail problems as if they imply that the basic idea of mathematics is misunderstood.

3. Logics in the argument. Questions often indicate that the students do not know what is going on, what is the goal of a calculation, i.e., the frames and purposes of calculations are unclear. The logical frames of argument are described more, and also in more non-formal ways.
4. Further explanations have been added concerning details in calculations.

4.3.3. *Questions that have not result in revisions*

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Some good questions do not result in revisions since they would lead too deep into the subject for a basic calculus course. In some cases a reference to another text is included.
We have a possible significant long term effect in that the author have recognized new aspects of the way students experience the mathematical material, which may lead to later revisions. Such recognition and reflection may lead to books with a different structure or basic idea – i.e. not only revisions.

4.4. *Scientific issues regarding project evaluation*

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An obvious scientific issue with the project evaluation is that it is evaluated by the project leader, who has both planned and has been responsible for the execution of the project. No external evaluator is involved. Furthermore, the project leader is also the author of the book. However, independent evaluations by students and answerers are included.
A positive scientific aspect is the openness of this project. It is open in the sense that it is possible to contact the project leader to receive further details about the project and the execution and performance thereof. Furthermore, the web site (in Swedish) www.bth.se/analysmeddialoger with existing dialogues is available for anyone to study.
Only 29 students participated in the project. A larger number of students would imply higher reliability in the questionnaire statistics.

4.5. *Comparison to goals*

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In Section 2.4, the following five questions have been formulated. We next discuss which answers or indications of answers to these questions have been provided by the project.

1. Which kind of improvements of a text book can students contribute with?

The statistics in the previous evaluation sections suggest the following types of improvements:

- Points of logical gaps in arguments.
- Further elaboration on generality of concepts, perhaps by including several diverse examples/applications.
- More examples of metaphors to connect abstract concepts to well known experiences.
- Handling of constants that appear in integration and in solving differential equations, a point that before the project has been regarded as a rather trivial non-issue by the project leader, as it is by the mathematics community.

Teachers who engage in dialogues with students can expect disclosures of this kind, thus extending the teacher's student body knowledge (see Section 2.2.1).

2. Can a text book be significantly improved by student feedback?

This is a question with a very large scope, as it depends on an even more basic question upon which teachers perhaps not always agree: what is a good text book?
As every student is different – in both learning ability and method – a good text book needs to provide different ways of explaining a given topic. It should be written so that it is easy to find ways to study mathematics in different ways, e.g., by problem solving, basic ideas, theorems and proofs, applications, or a combination of these.
Evidence of the project described in this report indicate that that long term student-teacher cooperation can effectively improve text books and educations. It opens a type of discussion between the author/teacher and students about the text book content that otherwise does not occur.

3. Is it possible to extend this feedback model to other subjects?

The fact that dialogues are less common in mathematics than in other subjects make them all the more important, but also more difficult. In most other subjects, dialogues can be expected to flow easier. Also, in most other subjects it is not necessary to find ways to write formulas with the usual text symbols. In the humanities and social sciences, the ability of verbal argumentation is considered to be part of the subject skill, which is reflected in the activities in these courses. A similar project, tailored more to the specifics of these areas, may give such activities more weight because of the potential changes in the text book. The idea of systematic student feedback is a fundamental idea which can be expected to be useful for text book improvement in any subject.

4. Does this project provide other benefits for the students' learning?

An impression while reading the student questions is that the process of formulating and posing questions fosters a more mature relation to mathematics. A common student attitude is to very strictly focus what is expected to be important for the examination, which may lead to a narrow knowledge of the subject. This may be a problem since it is difficult to construct examination that encourages a broad knowledge of the subject. In posing and answering questions, the student partly adopts a teacher role, which may develop a broader kind of knowledge.

Initially, many students have disliked posing questions. They have preferred to produce answers to specific questions before posing their own. Superficially, the former may seem easier. It is a common student habit of answering, rather than asking, questions – particularly in mathematics.

One point of the mandatory posing of questions is that it is a fairly reliable sign of activity. Questions can be hard to find, but once the mathematical work actually starts, questions of different kinds seem to appear. Having access to the web site, students can quickly and easily enter questions and answers – addressing a common problem for teachers of distance courses: to measure the activity of the students. Students who cannot produce questions are almost certainly passive in the course.

The fact that the questions of the students may alter a book read by future students may provide a sense of responsibility. This may increase the general responsibility for the education, as well as making mathematics a subject more open to questions.

Many questions have concerned the use of mathematical methods. Answers to these are important mathematics knowledge, as they strongly increase the ability to succeed in mathematical calculations and problem solving. Such method-related questions rarely appear in a dialogue-free learning environment.

Occasionally, students have exercised cooperation on the web site. Furthermore, in the evaluation students say that they did learn from reading other communications on the web site.

5. How much extra resources in time, personnel and equipment is necessary?

As is described by the answerer comment by David Erman and Maria Salomonsson above, answering questions involves a great deal of learning for the students involved. Students can use this as a teaching merit, and often their knowledge, interest and self-confidence in the subject increase. It should not be difficult to find students that to some extent may participate at a relatively low cost. It is also a means of involving talented students in department work.

Most answer typically takes from 1 to 10 minutes to complete. Some answers, though, require verifying long calculations or more involved information gathering, such as historical questions. If web communication is compared to helping students in a classroom, one can expect class room dialogues to be more effective in time. However, there are also advantages to web dialogues over class dialogues. First of all, web dialogues are naturally recorded, allowing them to affect the text book, and being available for other students. The timing is different, and better in some respects. It is possible to give more thought to an answer. In a class room, a student often accepts an argument that is not understood in order not to show incompetence. A teacher can in direct dialogue respond to a question before it is properly formulated. On the web, the student is forced to fully formulate a question, and has time to do this. Some students feel more comfortable in posing questions anonymously on a web site than in a class room. Both kinds of dialogues have specific advantages, and should be exercised.

When calculating the resources needed in time, web communication can be compared to classroom help. If students also are asked to answer questions, not only to pose them, the resources needed become smaller. Monitoring and completing dialogues becomes more important for the teacher, instead of answering to direct questions.

4.6. Conclusions and improvements

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The project described in this report is an attempt to enhance scientific maturity among students, improve text books in a long term perspective, and discuss which types of knowledge a teacher needs. Students are able to participate easily in both asking and replying to questions. A certain initial doubt regarding not only *answering* questions, but also posing them, is evident. Motivated and driven students who take their education seriously adapt faster to this new process. The project provides teachers (and authors) with better knowledge about students and their needs in acquiring (and retaining) the contents of a given course, resulting in improvement of both teaching and text books. Additionally, the project allows students to exercise several competences: formulation of arguments and scientific questions, student responsibility and cooperation.

By feedback activity, education and text books evolve by the education and teaching processes. A feedback project breaks the potential isolation between teachers and students by encouraging or enforcing student participation, thus encouraging the development of the students' scientific maturity.

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